Existence, Convergence and Efficiency Analysis of Nash Equilibrium and Its Application to Traffic Networks

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(Joint work with Xuehe Wang)

Outline

- **Motivation**
- **Related Work**

- **Road Pricing Strategies: A Nash Equilibrium Perspective**

  - Distributed consensus in non-cooperative games
  - Routing problem
  - Price of anarchy

- **Conclusion and Opportunities**

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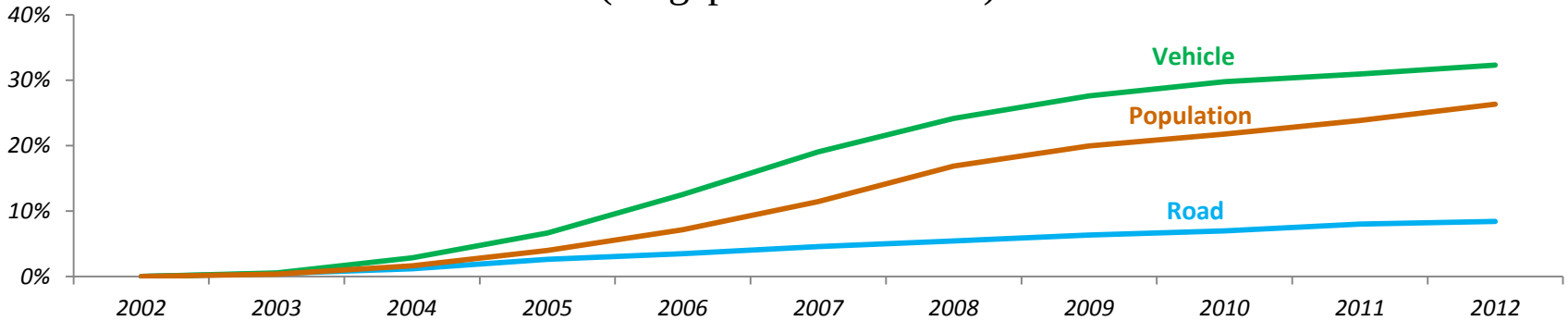
Motivation

Rapid Vehicle & Population Growth vs ***Limited*** Road Development
(Singapore 2002~2012)



Source: Land Transport Authority of Singapore

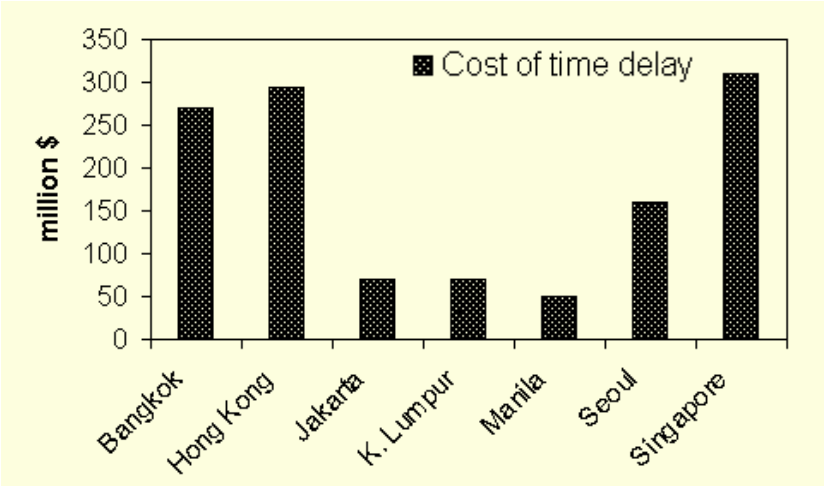


	Vehicle	Population	Road
2012	0.97 million	5.3 million	3,425 km
2030	1.2 million	6.9 million	3,700 km
Increase	~24%	~30%	~8%

Traffic control makes full utilization of existing infrastructure without road expansion

Motivation

- Traffic Congestion Problem
  - Traffic congestion causes significant efficiency losses, wasteful energy consumption, and excessive air pollution
  - It is difficult to enlarge the roadway capacity in major urban areas

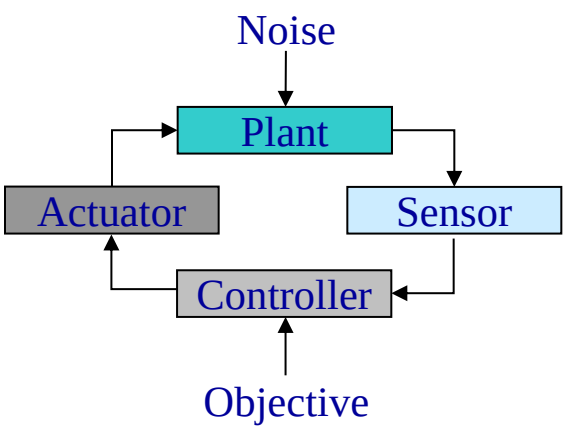


Relationship between economic losses and traffic congestions in key South-East Asian cities. World Resources Institute, *World Resources 1996-97*, 1997

- Congestion Losses
  - In Europe, the external costs of road traffic congestion amount to 0.5% of Community GDP (White Paper—European Transport Policy for 2010)
  - Sri Lanka loses 1.5% of the GDP (Rs 32 billion) due to traffic congestion (Business Times 2011)
  - In UK, traffic congestion is costing the economy more than GBP 4.3 billion a year (Centre for Economics and Business Research 2012)

Traffic Control Systems

Basic Control system

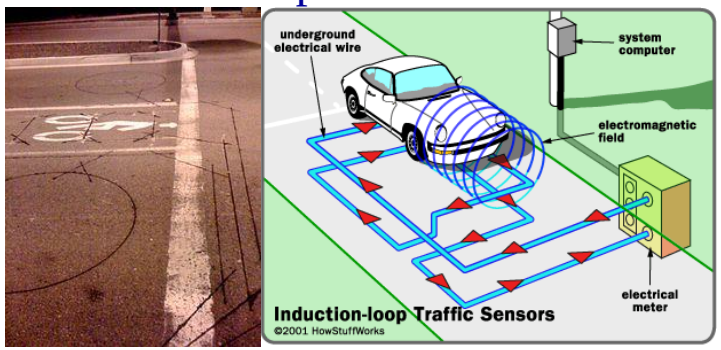


Component	Illustration
Plant	Dynamics of traffic flows
Sensor	(1) Loop detectors (2) Pneumatic tube counter (3) Cameras
Controller	(1) Road price (2) Traffic signals (3) Variable-message sign
Actuator	Driver's action and/or decision
Noise	(1) Demand variations (2) Accidents, raining, fog
Objective	(1) Minimization of travel delay and/or number of stops (optimized performance) (2) Boundedness of vehicle queues (stability)

A control system to establish traffic regulations and their communications to the driver

Traffic Sensors

Loop Detector



Camera

Pneumatic Tube Counter

Traffic Controllers

Road Pricing

Traffic Signals

Variable-message Sign

Other controllers

- Signs and markings
- Car pooling programs
- Number plate auction

ERP in Singapore

- Electronic Road Pricing (ERP System)

  Alleviate traffic congestion by

  - affecting road users' routing choices
  - refraining road users from travelling during peak hours

ERP gantry in Singapore

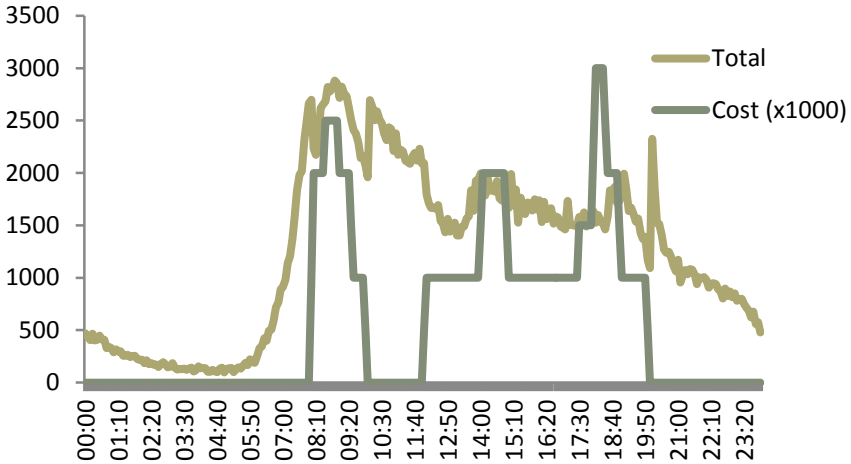
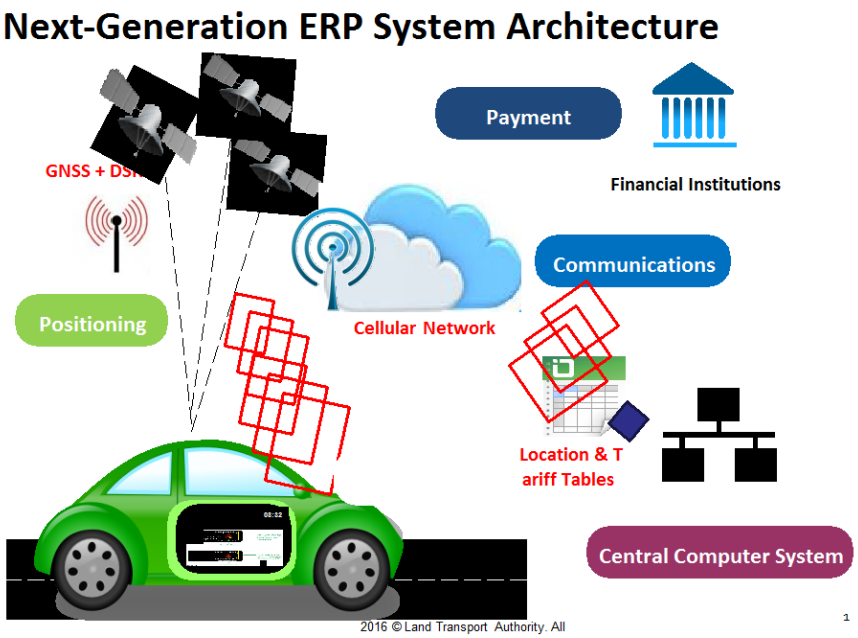


Fig. 1 The effect of ERP to number of vehicles on Anson Road in Singapore

- Effect of Road Pricing (ERP system)
  - Traffic flow shoots up quickly just before the ERP begins
  - Sharp decrease just at the time ERP begins
  - Sharp rise just after ERP ends

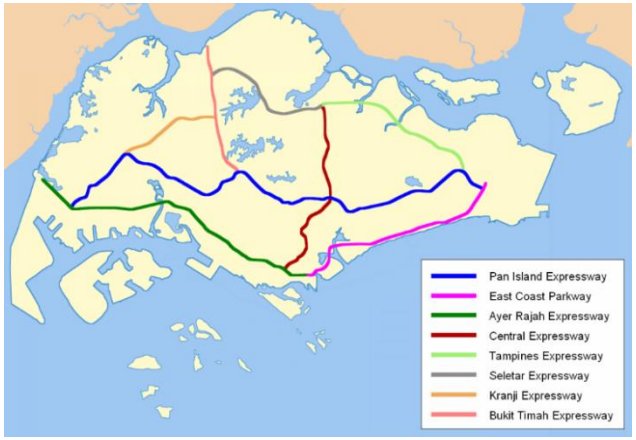
ERP2 in Singapore

- Expected to be implemented progressively from 2020 by the consortium of NCS Pte Ltd and Mitsubishi Heavy Industries Engine System Asia Pte Ltd, at a cost of S$556 million
- Based on Global Navigation Satellite System (GNSS) and Dedicated Short Range Communication (DSRC)
- Allow for more flexibility in managing traffic congestion through distance-based road pricing
- Provide services for motorists' convenience, such as disseminating information on traffic advisories and facilitating e-payments



Objective

- Traffic Network



  - Routing Problem: Multiple origin-destination pairs and each origin-destination pair has several routes
  - Trip Timing Problem: Different departure time to avoid peak hour

- **Objective**: To develop road pricing strategies based on game theory and consensus control to manage traffic flows in an optimal manner and minimize traffic congestion

  - To estimate the mass behavior of all players via consensus control
  - To analyze the efficiency of Nash equilibrium and the performance in evolution of repeated games
  - To design dynamic pricing control to improve the efficiency of Nash equilibrium and the overall efficiency in repeated games for trip timing and routing problem

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Related Work

➢ Game theory: deals with strategic interactions among multiple players, where each player tries to maximize his/her own utility (R. Gibbons, 1992).

Elements of a game

- Players: road users
- Strategies: route choices or trip timing available to each player i, $s_i \in R$
- Utility function: $U_i(s)$,where $s = (s_i, s_{-i})$

Equilibrium concept (Nash)

- $s^{ne} = (s_i^{ne}, s_{-i}^{ne})$ is a Nash equilibrium if for any player i,

$$U_i(s_i^{ne}, s_{-i}^{ne}) = \max_{s_i \in R} U_i(s_i, s_{-i}^{ne})$$

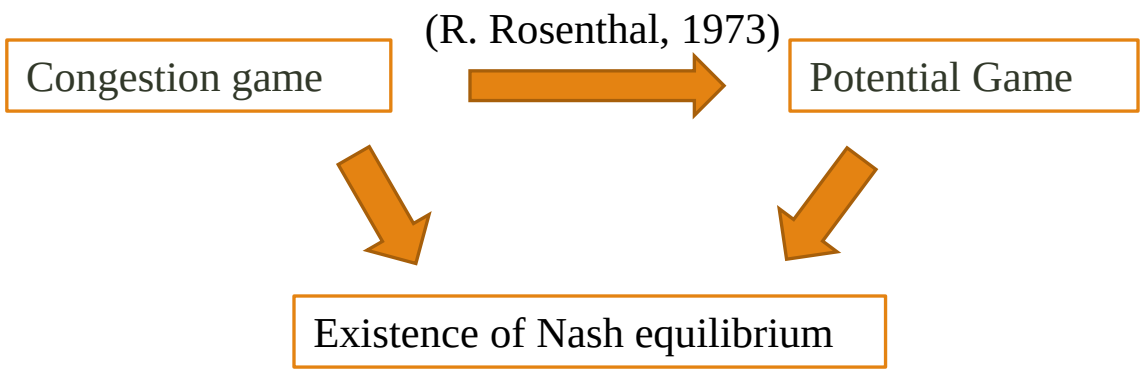
- None of players can improve his/her utility by a unilateral move
- Not always exist and may be inefficient

Related Work

- **Congestion game:** The utility/cost of each player depends on the strategy/resource it chooses and the number of players choosing the same strategy/resource (R. Rosenthal, 1973).
- **Potential game:**

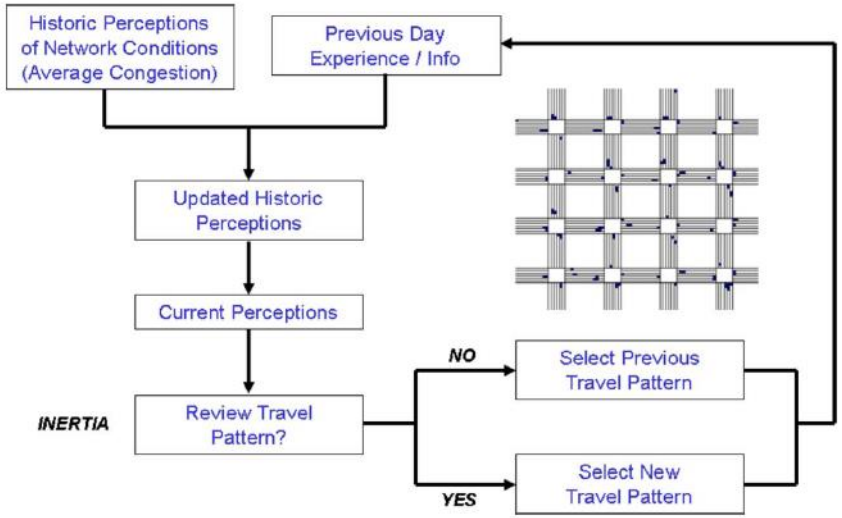
$$U_i(s_i^1, s_{-i}) - U_i(s_i^2, s_{-i}) = \Phi(s_i^1, s_{-i}) - \Phi(s_i^2, s_{-i})$$

  - Guarantee the existence of Nash equilibrium (D. Monderer et al, 1996).
  - All players tend to jointly optimize the potential function.



Related Work

- **Learning in games:** Allows players to adapt their strategies in response to the available information gathered over prior stages (D. Fudenberg et al, 1998).
  - Update perceptions of traffic conditions based on information broadcasted by government and/or obtained from other drivers through V2X
  - Inertia (intuitively): Some reluctance to change previous travel pattern



Driver's decision process (from J.R. Marden etc. 2009)

Learning of Games: Fictitious Play

- Complete information of all utility functions is generally not available for individual player

- (Monderer 1996) Fictitious play: each player assumes that other players make decision independently according to observed empirical frequencies. The empirical frequencies generated by fictitious play of a potential game converge to a mixed strategy NE

- Shortcoming: when number of players is large, actual action for player *i* at every stage is computationally infeasible since it depends on a mapping over a joint space

- (Marden et al. 2009) Joint strategy fictitious play: each player assumes that other players make decisions randomly and jointly according to joint empirical frequencies. It still ensures the convergence for potential games and reduces the computational burden of standard fictitious play

- Shortcoming: utility updating process is required for each player at every stage

Related Work

- **Discrete-time consensus protocol**: To estimate the number of players choosing each strategy for binary strategies case in inventory games (D. Bauso et al, 2009).
  - Exchange information with a set of neighbors.
  - Initial state of each agent – initial strategy of each player.
  - Global objective: Average-consensus – the percentage of players choosing each strategy.
- **Pricing schemes:** To improve the efficiency of Nash equilibrium.
  - The first-best pricing (marginal-cost pricing): the difference between the marginal social cost and the marginal private cost (M. J. Beckmann, 1967; M. Smith, 1979; H. Yang et al, 2004).
  - The second-best pricing: the location of the toll-gate, how much to charge, and the different impacts of the pricing schemes on different users (M. Marchand, 1968; T. Tsekeris et al, 2009).
  - Dynamic road pricing: vary according to real time road condition (R. B. Dial, 1999; T. Wongpiromsarn et al, 2012).

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Distributed Consensus in Non-cooperative Congestion Games

- **Model Setup**

Public transportation or private car and departure time?

- **Notations:**
  - $\mathcal{R}_1 = \{r_1, r_2, ..., r_v\}$: public transportations.
  - $\mathcal{R}_2 = \{r_{v+1}, r_{v+2}, ..., r_{v+m}\}$: trip timing choices.
  - $s_i(k) = [s_{i1}(k), ..., s_{iv}(k), s_{iv+1}..., s_{iv+m}(k)]^T \in \mathbb{Z}^{v+m}$: player i's choices,

    where $s_{ij} \in \{0, 1\}$ with $\sum_{j=1}^{v+m} s_{ij} = 1$.
  - $s(k) = \{s_1(k), s_2(k), ..., s_N(k)\}$: the action profile of all players.

Distributed Consensus in Non-cooperative Congestion Games

- **Problem Formulation – Trip Timing Problem**
  - Utility of player i:

$$U_i(s_i(k), s_{-i}(k)) = \sum_{j=1}^{v} s_{ij}(k)p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(k)(f_j(n_{r_j}(k)) + h_i(r_j)) \quad (1)$$

public transportation

Utility due to congestion

fixed utility

- **Lemma 1**: Congestion game with utility function (1) is a potential game with potential function:

$$\phi(s) = \sum_{j=1}^{v}\sum_{i=1}^{N} s_{ij}p_{ij} + \sum_{j=v+1}^{v+m}\sum_{d=1}^{n_{T_j}} f_j(d) + \sum_{j=v+1}^{v+m}\sum_{i=1}^{N} s_{ij}h_i(T_j)$$

Existence of Nash equilibrium

Distributed Consensus in Non-cooperative Congestion Games

- If each player knows others' strategies, $\boldsymbol{min_{s_i(k)} U_i(s_i(\mathrm{k}), s_{-i}(\mathrm{k}))}$
- **Consensus Protocol**

$x_i(k) \in \mathbb{R}^{v+m} \quad z_i(k) \in \mathbb{R}^{v+m}$

$\alpha(k) \in (-1/\Delta(k), 0),$
$\Delta(k) = \max_i(L_{ii}(k))$

Pre-decision information

$$x_i(k+1) = z_i(k) + \alpha(k) \sum_{j \in N_i(k)} L_{ij}(k) z_j(k)$$

$$z_i(k) = x_i(k) + s_i(k) - s_i(k-1) \ \forall k \geq 1$$

$$z_i(0) = x_i(0) = s_i(0)$$

Post-decision information

i, j entry of Laplacian matrix

Neighbor set

Consensus protocol for multiple strategies case: alleviate the binary constraint of the inventory game (D. Bauso et al, 2009).

- **Lemma 2:** If no player changes strategy from stage t on and the collection of graphs over $\tilde{T}$ is jointly connected, then $x_i(\mathrm{t} + \tilde{T})$ is an estimate of the percentage of players choosing each resource at stage t.

Wang et al, Decentralized Dynamic Games for Large Population Stochastic Multi-Agent Systems," IET Control Theory and Application, Vol. 9, No.3, pp. 503-510, Feb. 2015.

Estimate Percentage of Players with Broadcasting

- Step 1: at initial stage $t = 0$, every player picks up an action arbitrarily.
- Step 2: at stage $t \geq 1$, a system supervisor records $n_{r_l}(x(t-1)),\ l = 1, 2, \ldots, M,$ and computes its weighted running average recursively as

$$\bar{n}_{r_l}(0) = n_{r_l}(x(0)),$$
$$\bar{n}_{r_l}(t) = (1-\lambda)\bar{n}_{r_l}(t-1) + \lambda n_{r_l}(x(t-1)),$$

where $\lambda \in (0, 1]$.

- Step 3: $\bar{n}_{r_l}(t)$ is broadcasted by system supervisor to all players, and player uses it for action selection

Distributed Consensus in Non-cooperative Congestion Games

➢ Convergence of Nash Equilibrium

- **Inertia:**
  - ✓ If no opportunity for utility improvement, stay with previous strategy, i.e., $s_i(k) = s_i(k-1)$
  - ✓ Otherwise, choose strategy which has maximal utility with probability $\theta_i(k)$,where $0 < \theta_i(k) < 1$
- **Theorem 1**:

$\mathcal{G}_N(k)$: jointly connected over each time interval $\tilde{T}$

maintain strategies

Inertia

$U_i(s_i^1, s_{-i}) \neq U_i(s_i^2, s_{-i})$

Consensus protocol

Converge to Nash equilibrium almost surely

Distributed Consensus in Non-cooperative Congestion Games

➢ Dynamic Pricing

Marginal-cost (Pigovian tax, Mankiw, 2009) :

$$p_j(n_{r_j}) = (n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j}))$$

- Charge when players enter the road
- Make players aware of the social cost instead of private cost

Utility of player i:

$$U_i(s_i(k), s_{-i}(k)) = \sum_{j=1}^{v} s_{ij}(k)p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(k)(f_j(n_{r_j}(k)) + h_i(r_j) - p_j(n_{r_j}(k)))$$

Potential function:

$$\bar{\phi}(s) = \sum_{i=1}^{N}(\sum_{j=1}^{v} s_{ij}p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(f_j(n_{r_j}) + h_i(r_j)))$$

Some sort of social optimal

- All players jointly optimize the social utility

Distributed Consensus in Non-cooperative Congestion Games

➢ Case Study of Singapore

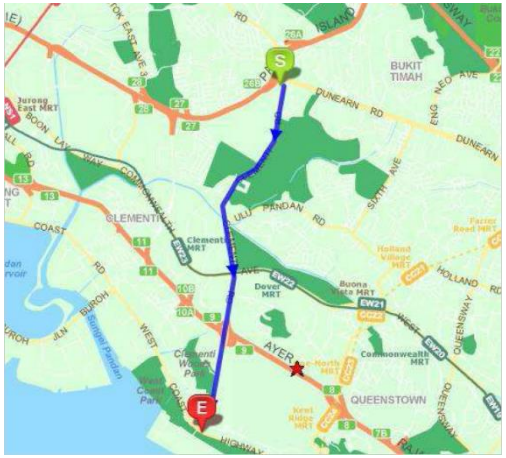
- Travel time formula introduced by P. Patriksson, 1994:

$$d(n_{r_j}) = t_0 \cdot (1 + 0.15(n_{r_j}/q_0)^{m_\alpha})$$

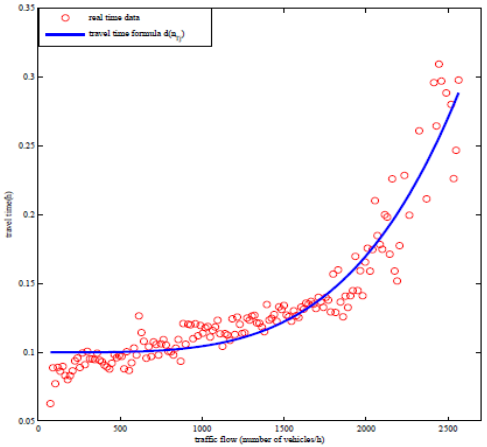
Positive parameter

Free-flow travel time: travel time at zero flow

Practical capacity: the ow from which the travel time will increase very rapidly if the ow is further increased



Clementi Road (length 5.4 km)



Data fitting:
$t_0 = 0.0998$,
$m_\alpha = 3.977$,
$q_0 = 1357$

Fig .2 The relationship between traffic flow and travel time

Distributed Consensus in Non-cooperative Congestion Games

➢ Case Study of Singapore

$N = 2000,\ \alpha_i \in [-1.5, -0.5]$

- Average speed: $f_j(n_{r_j}) = \dfrac{L_0}{d(n_{r_j})}$
- Deviation from preferred departure time: $h_i(r_j) = \alpha_i |r_j - \hat{T}_i|$

Overall utility without price: 95219
Overall utility with price: 96781

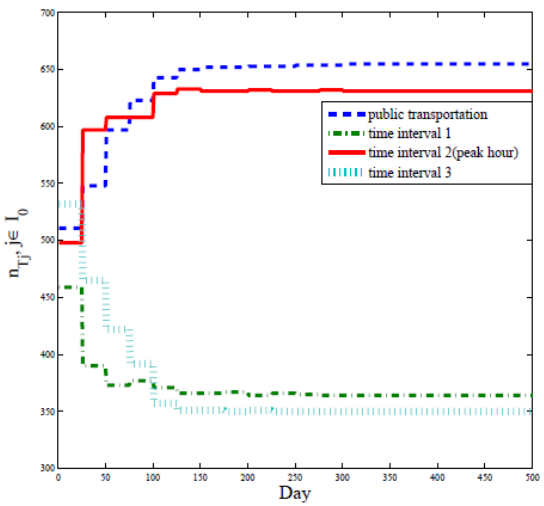
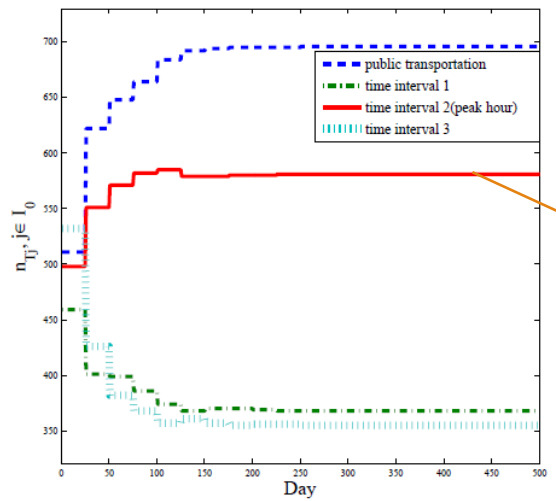


Fig. 3 Evolution of number of vehicles on each choice without price.



May still cause traffic congestion

Fig. 4 Evolution of number of vehicles on each choice with price.

Distributed Consensus in Non-cooperative Congestion Games

➢ Dynamic Pricing

- **Case without Public Transportation**

Entropy term: $\sum_{j=v+1}^{v+m} q_j \log q_j$

Percentage of players choosing resource j

- Reach minimum value only when $q_{v+1} = \dots = q_{v+m}$

Road pricing:

$$\tilde{p}_j(n_{r_j}) = (n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j}) + \frac{w}{N} log(\frac{n_{r_j}}{n_{r_j} - 1})) + \frac{w}{N} log(n_{r_j})$$

Potential function:

$$\tilde{\phi}(s) = \sum_{i=1}^{N} (\sum_{j=v+1}^{v+m} s_{ij}(f_j(n_{r_j}) + h_i(r_j))) - w \sum_{j=v+1}^{v+m} q_j \log q_j$$

Spread out players' strategies

- Action profiles generated by the utility function converges to pure NE a.s.
- As w goes to infinity, the disparity in the number of players choosing each resource will vanish

Distributed Consensus in Non-cooperative Congestion Games

- **Case Study of Singapore**

  - **Case without Public Transportation**

Set w=200000

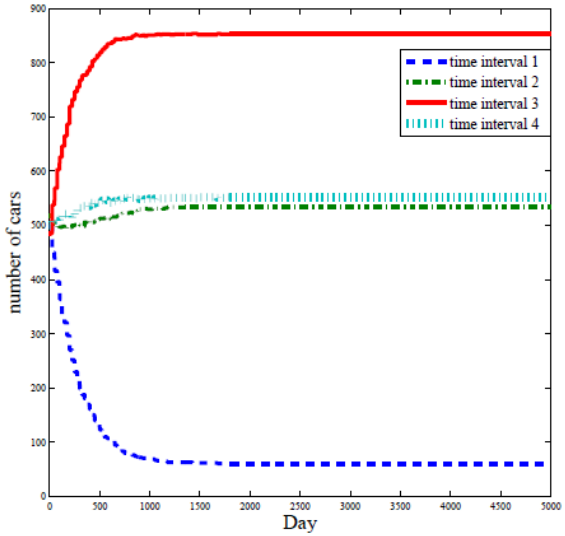


Fig. 5 Evolution of number of vehicles on each choice without price

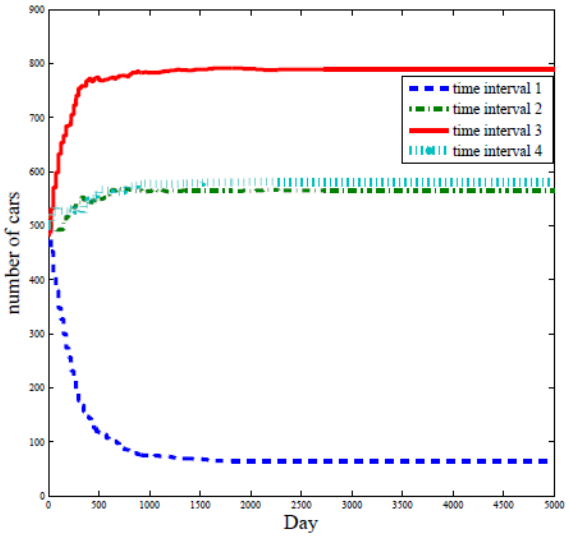


Fig. 6 Evolution of number of vehicles on each choice with marginal cost

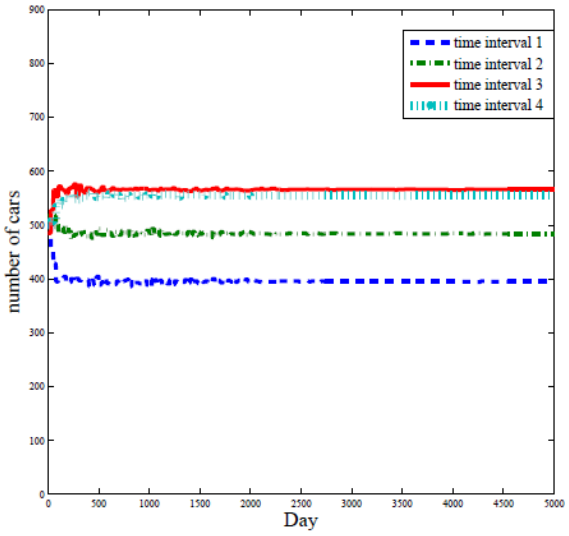


Fig. 7 Evolution of number of vehicles on each choice with price function with entropy term

Road Pricing for Routing Problem

- **Price Sensitivity**

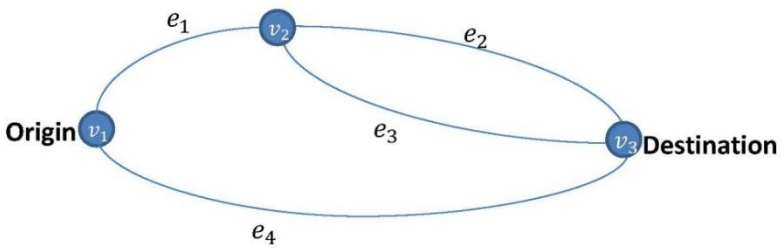
Each road user has a price sensitivity β which may be different for different road users

- $\{\beta_1, \ldots, \beta_M\}$: set of price sensitivities
- $\{p_1, \ldots, p_M\}$: the corresponding distribution

Mode	Value of Time (¢/min)	Price sensitivities (min/¢)
Car	4.0	0.25
Motorcycle	2.8	0.36
Taxi	5.1	0.20

Price sensitivities (from 2004 Stated Preference Survey)

Road Pricing for Routing Problem



- Traffic network
  - Each edge e possesses a linear latency function (T. Roughgarden et al, 2002) :

$$l_e(f_e) = d_e f_e + c_e$$

- d_e: additional travel time due to one more player on link e
- c_e: free-flow travel time on link e

  - Latency of each route r:

$$l_r(f_r) = \sum_{e \in r} l_e(f_e)$$

  - Total latency of the network:

$$L(f) = \sum_{r \in \mathcal{R}} f_r \cdot l_r(f_r)$$

Social optimal flow:
$f^* = \arg\inf_f L(f)$

  - Cost of route r for group with price sensitivity β:

$$J_{r,\beta}(f) = \sum_{e \in r} (l_e(f_e) + \beta \rho_e(f_e))$$

$\rho_e(f_e)$: Toll on each edge e

Wang et al., Analysis of Price of Anarchy in Traffic Networks with Heterogeneous Price-sensitivity Populations, IEEE Transactions on Control System Technology, Vol. 23, No. 6, pp. 2227-2237, 2015.

Road Pricing for Routing Problem

- Nash flow (Nash equilibrium): A flow f^{ne} is called a Nash flow, if for any β_j and r_1, r_2,

$$f^{ne}_{r_1,\beta_j} > 0, f^{ne}_{r_2,\beta_j} > 0 \Rightarrow J_{r_1,\beta_j}(f^{ne}) = J_{r_2,\beta_j}(f^{ne}),$$
$$f^{ne}_{r_1,\beta_j} > 0, f^{ne}_{r_2,\beta_j} = 0 \Rightarrow J_{r_1,\beta_j}(f^{ne}) \leq J_{r_2,\beta_j}(f^{ne}).$$

May be inefficient!

  - All players choose minimum-cost routes
  - For each group, the costs of the selected routes are the same

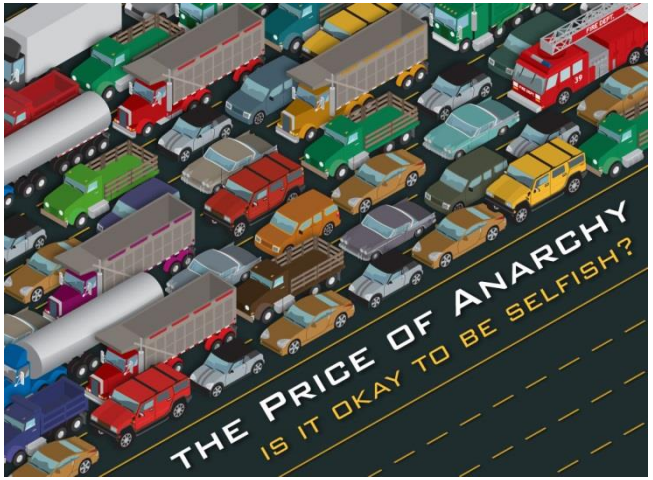
- Price of anarchy (POA): The worst possible ratio between the total latency of a Nash flow and that of the optimal flow:

$$\text{POA} = \sup_{f^{ne}} \frac{L(f^{ne})}{L(f^*)}$$

POA: the smaller, the better

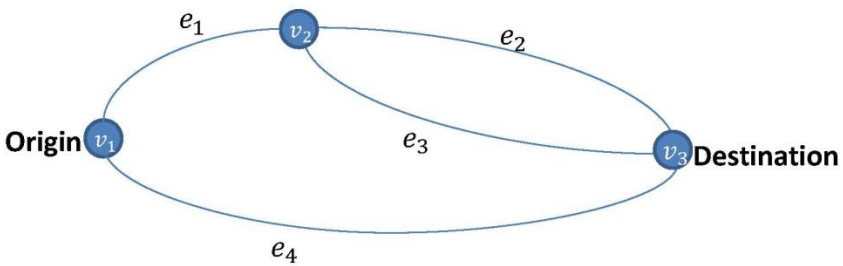
- The POA can be arbitrarily large
- It is proved by T. Roughgarden et al. (2002) that POA is bounded by 4/3 if the network has linear latency functions
- Sensitive to users' price sensitivities (Brown and Marden, CDC 2014). Nash flow could be as much as 33% more congested than an optimal flow for unknown sensitivities.

Road Pricing for Routing Problem



- **Method of Reducing POA**
  - Marginal-cost: The difference between the marginal social cost and the marginal private cost
    - ✓ Make player aware of the losses that it imposes on other players
    - ✓ Totally eliminate the efficiency losses due to selfish behavior of players, i.e., POA=1
    - ✓ Only apply to homogeneous case when players have the same price sensitivity
  - Scaled marginal cost: A variant of marginal cost which can be applied to heterogeneous case

Road Pricing for Routing Problem



➢ Design of Road Pricing

Same price sensitivity β

Homogeneous players

Heterogeneous players

Marginal cost toll on each edge e

$$\rho_e(f_e) = f_e l'_e(f_e)$$

Scaled marginal cost toll on edge e

$$\rho_e(f_e) = \mu f_e l'_e(f_e)$$

Nash flow ⟹ Social optimal flow

Goal: $L(f^{ne}(\mu^*)) = \inf_{\mu \geq 0} L(f^{ne}(\mu))$

Remark: Homogeneous case with uncertain $\beta \in [\beta_L, \beta_U]$, $\mu^* = \frac{1}{\sqrt{\sqrt{\beta_L \beta_U}}}$ (Brown and Marden, 2014)

Wang et al., Analysis of Price of Anarchy in Traffic Networks with Heterogeneous Price-sensitivity Populations, IEEE Transactions on Control System Technology, Vol. 23, No. 6, pp. 2227-2237, 2015.

Road Pricing for Routing Problem

➢ Analyze of Price of Anarchy

- **Theorem 2**: The optimal POA depends on the distribution of price sensitivities and the topology of the network.

At most one group β_j chooses more than one routes

Distribution of β satisfies certain conditions

$\mu^* = 1/\beta_j$ such that POA=1

At least two groups chooses more than one routes

Distribution of β doesn't satisfy certain conditions

μ^* such that POA>1

- Special case – Two groups of players in a two routes network: always exists μ^* such that POA=1
- POA depends on the distribution of price sensitivity, road network topology and parameters of latency functions.

Road Pricing for Routing Problem

- **Analyze of Price of Anarchy**

  - Best POA Algorithm: find μ^* that minimizes the POA for any distribution of price sensitivity and networks with one origin-destination pair

Remarks:

- The results on the non-cooperative games can be extended to include combined routing and trip timing
- Can be extended to multiple S-D case.

Input: Network parameters (e.g., d_e, c_e) and network topology, total flow F, price sensitivities $\mathcal{B}$ and its probability distribution P
Output: Best POA
Calculate the social optimal flow f^*;
for $k_1 = 1 : N$
  Group β_1 choose a sequential routes $r_{k_1}, ..., r_N$ with flow $f_{r_{k_1},\beta_1} > 0, ..., f_{r_N,\beta_1} > 0$;
  $\vdots$
  for $k_{j+1} = 1 : k_j$
    Group β_{j+1} choose a sequential routes $r_{k_{j+1}}, ..., r_{k_j}$ with flow $f_{r_{k_{j+1}},\beta_{j+1}} > 0, ..., f_{r_{k_j},\beta_{j+1}} \geq 0$;
    $\vdots$
    for $k_M = k_{M-1} - 1 : k_{M-1}$
      Group β_M choose a sequential routes $r_1, ..., r_{k_M}$ with flow $f_{r_1,\beta_M} > 0, ..., f_{r_{k_M},\beta_M} > 0$;
        Solve the equations set
        Return Nash flow $f^{ne}(\mu)$;
      Find μ^* by optimizing $L(f^{ne}(\mu))$;
    end
    $\vdots$
  end
  $\vdots$
end
Return the largest value of $\frac{L(f^{ne}(\mu^*))}{L(f^*)}$

Road Pricing for Routing Problem

- **Real Data Simulation**

4 groups of road users with total flow 200.

Group	Price sensitivity(min/¢)
Car	0.25
Motorcycle	0.36
Taxi	0.20
Bicycle	0

Price sensitivities for each group (from 2004 Stated Preference Survey)

Group	Vehicle Distribution
Car	0.7239
Motorcycle	0.1884
Taxi	0.0281
Bicycle	0.0596

Vehicle distribution (from Singapore land transport statistics in brief 2005)

Road Pricing for Routing Problem

➢ Real Data Simulation

Consider the East Road Network of Singapore with 4 routes

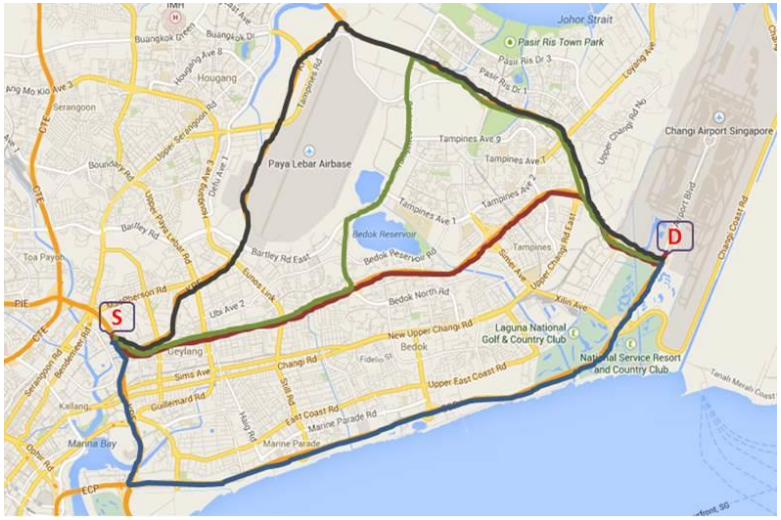


Fig. 8 Traffic network in the east of Singapore

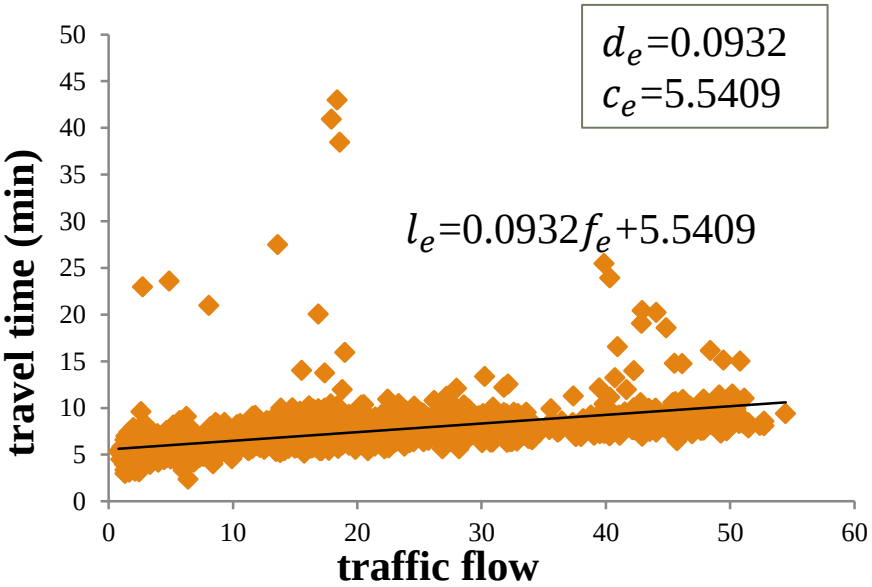


Fig. 9 Linear fitting of certain link

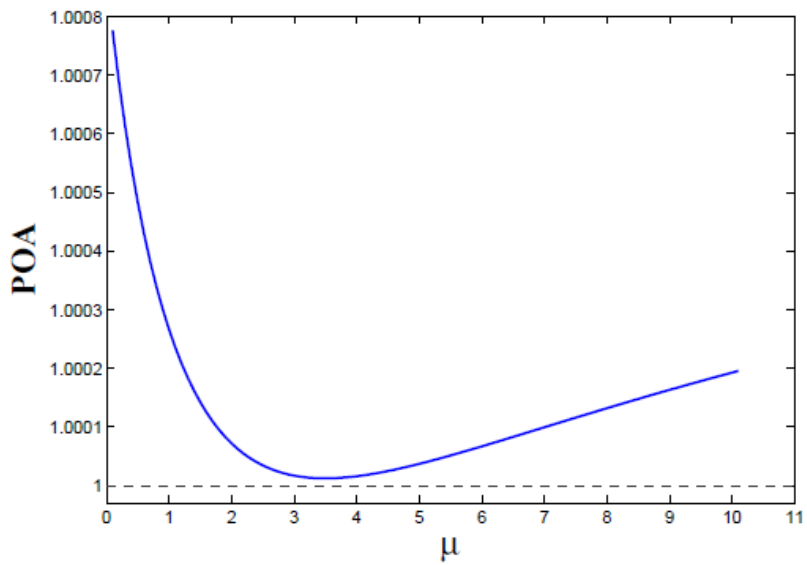
Road Pricing for Routing Problem

➢ Real Data Simulation

Optimal road price: $\mu^* = 3.394$ such that POA>1, i.e., the total latency of the Nash flow can't achieve the total latency of the socially optimal flow.

- Total latency without road price: 35950.711
- Total latency with road price: 35922.899
- Total latency of the optimal flow: 35922.807

Social welfare increases



POA deviates slightly from 1 at $\mu^* = 3.394$

Fig. 10 The relationship between μ^* and POA for road network

Road Pricing for Routing Problem

➢ Real Data Simulation

Consider the road network with 3 routes in CBD.

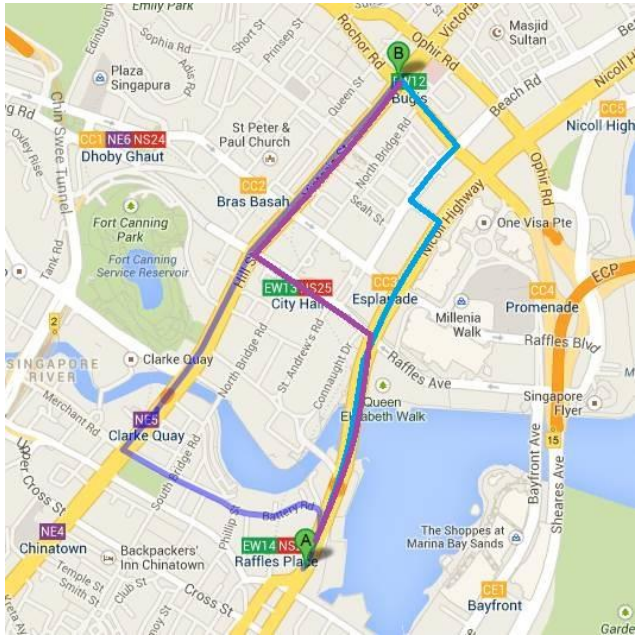


Fig. 11 Simple traffic network in Central Business District (CBD)

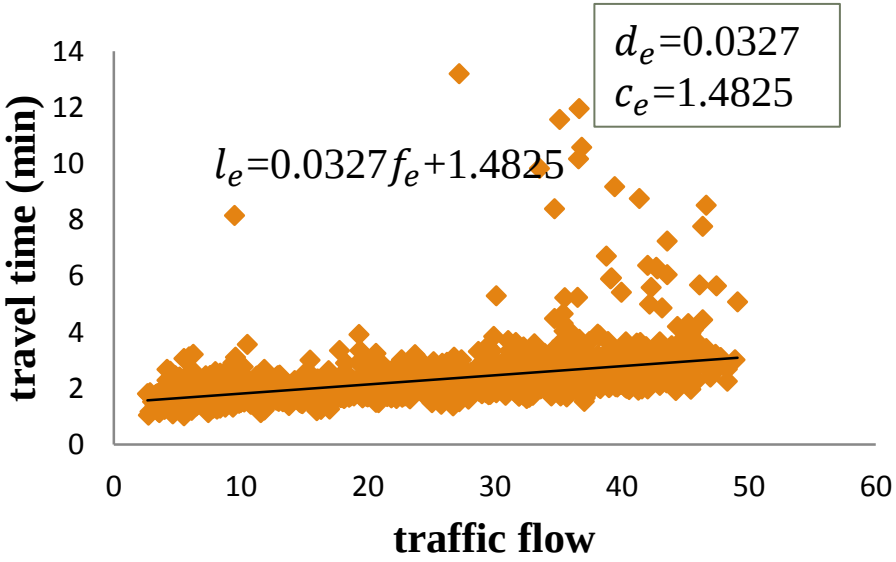


Fig. 12 Linear fitting of certain link

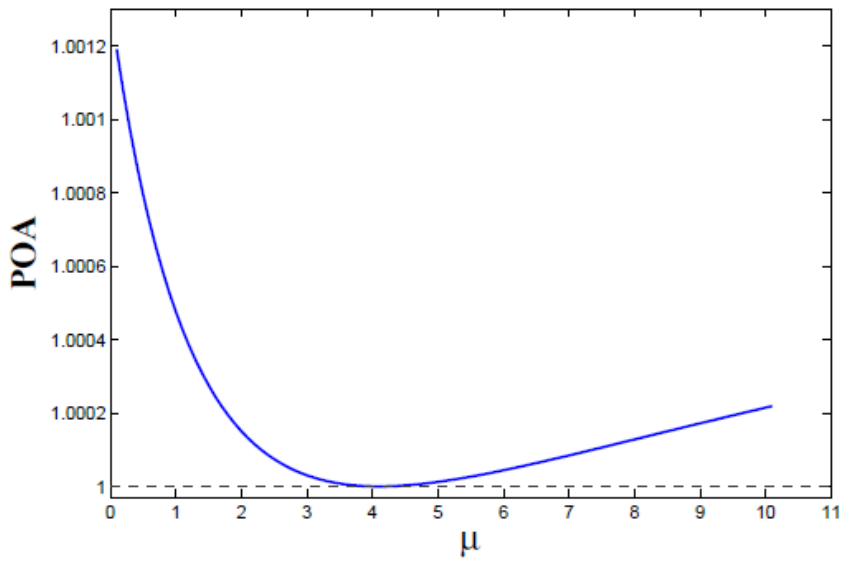
Road Pricing for Routing Problem

➢ Real Data Simulation

Optimal road price: $\mu^* = 4$ such that POA=1, i.e., the total latency of the Nash flow is equal to the total latency of the socially optimal flow.

- Total latency without road price is 2472.8.
- Total latency with road price is 2457.8.

Social welfare increases



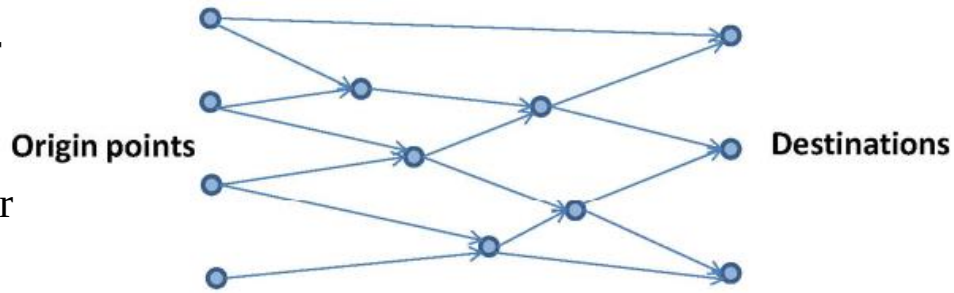
POA reaches 1 at $\mu^* = 4$

Fig. 13 The relationship between μ^* and POA for road network

Analysis of Price of Total Anarchy

➢ Problem Formulation

- Each origin-destination pair: fixed number of players
- Each player i selects a route s_i from his/her strategy set S_i (players in same origin-destination pair have the same strategy set)



- Latency $l_e(f_e)$ on each edge is associated with the number of players on this edge
- Cost received by player i is

$$C_i(s) = \sum_{e \in s_i} l_e(f_e)$$

- Nash equilibrium: for each player i,

$$C_i(s^{ne}) = \min_{s_i \in S_i} C_i(s_i, s_{-i}^{ne})$$

May be inefficient

Congestion game: Nash equilibrium always exists

Wang et al, Analysis of Price of Total Anarchy in Congestion Games via Smoothness Arguments, IEEE Tran. Control of Network Systems, to appear.

Analysis of Price of Total Anarchy

➢ No-regret

- Consider a sequence of strategies $s(1), s(2), \ldots, s(T)$, define

$$\delta_i(s(t)) = C_i(s(t)) - \min_{s_i \in S_i} C_i(s_i, s_{-i}(t))$$

- A strategy sequence exhibits almost sure ε-no-regret iff

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \delta_i(s(t)) < \varepsilon \qquad \text{a.s.}$$

If the sequence exhibits almost surely ε-no-regret for any $\varepsilon > 0$, then it is said to exhibit almost sure no-regret.

- For strategy profiles generated by best response with inertia, there exists a $\varepsilon_T > 0$ such that

$$\frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \delta_i(s(t)) = \frac{1}{T} \sum_{t=1}^{T} \left(\sum_{i=1}^{N} C_i(s(t)) - \sum_{i=1}^{N} \min_{s_i \in S_i} C_i(s_i, s_{-i}(t)) \right) \leq \varepsilon_T$$

$\varepsilon_T \to 0$ as $T \to \infty$ ⇒ Almost sure no-regret

a.s.

Analysis of Price of Total Anarchy

➢ Price of Total Anarchy

- The price of total anarchy (POTA) is defined as the worst-ratio between the average total latency and the total latency of the optimal assignment

$$\text{POTA} = \frac{\frac{1}{T}\sum_{t=1}^{T} C(s(t))}{C(s^*)}$$

- A cost-minimization game is (λ, μ)-smooth if for every two strategies s^1 and s^2

$$\sum_{i=1}^{N} C_i(s_i^2, s_{-i}^1) \leq \lambda \cdot C(s^2) + \mu \cdot C(s^1) \tag{2}$$

Specially, for a Nash equilibrium s^{ne} and the optimal strategy s^*

$$C(s^{ne}) = \sum_{i=1}^{N} C_i(s^{ne}) \leq \sum_{i=1}^{N} C_i(s_i^*, s_{-i}^{ne}) \leq \lambda \cdot C(s^*) + \mu \cdot C(s^{ne})$$

If a game is (λ, μ)-smooth with $\lambda>0$ and $\mu<1$, then POA $\leq \frac{\lambda}{1-\mu}$.

Analysis of Price of Total Anarchy

➢ Price of Total Anarchy

- Define $\delta_i^*(s(t)) = C_i(s(t)) - C_i(s_i^*, s_{-i}(t))$, obviously, $\delta_i(s(t)) \geq \delta_i^*(s(t))$

- In smooth game, at stage t,

Almost sure no-regret property due to

$$\frac{1}{T}\sum_{t=1}^{T} C(s(t)) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{1}{1-\mu}\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N} \delta_i^*(s(t))$$

- The robust POA of a cost-minimization game is

$$\inf\{\frac{\lambda}{1-\mu} : (\lambda, \mu) \text{ such that the game is } (\lambda, \mu) - \text{smooth}\} \quad \mu < 1$$

.

Theorem 3: The action profiles generated by best response with inertia will converge a NE a.s. in finite time and if the robust POA of the game is ρ, then there exists a $\varepsilon_T > 0$ such that

$$\text{POTA} \leq \rho + \frac{\varepsilon_T}{(1-\mu)C(s^*)}$$

almost surely, where $\varepsilon_T \to 0$ as $T \to \infty$

Analysis of Price of Total Anarchy

➢ Traffic Network with Linear Latency

Set the latency on each edge as $l_e(f_e) = a_e f_e + b_e$

- The networks with linear latency functions is a $(\frac{5}{3}, \frac{1}{3})$-smooth game (G. Christodoulou et al, 2006)

Lemma 3: For action profiles generated by best response with inertia,
POTA $\leq \frac{5}{2} + \frac{3\varepsilon_T}{2C(s^*)}$ almost surely, where $\varepsilon_T \to 0$ as $T \to \infty$.

- To improve the overall efficiency, motivated by the form of marginal-cost price, the road price is designed as

$$p_e(f_e) = w_e(f_e - 1)$$

w_e : Nonnegative constant to be designed

Lemma 4: For action profiles generated by best response with inertia and $w_e^* = a_e$
POTA $\leq \frac{49}{24} + \frac{\tau_T}{C(s^*)}$ almost surely, where $\tau_T \to 0$ as $T \to \infty$.

Remark: By charging the road price $p_e(f_e) = a_e(f_e - 1)$, the upper bound of the POTA decreases compared to that without road price

Analysis of Price of Total Anarchy

➢ Traffic Network with Nonlinear Latency

- Latency on each edge (Patriksson, 1994)

$$l_e(f_e) = t_e \cdot (1 + d_e(f_e/c_e)^{m_e})$$

Lemma 5: For congestion games with nonlinear latency, if the action profiles are generated by best response with inertia, then,

$$\text{POTA} \leq \frac{(\vartheta+1)^{2m_0+1} - \vartheta^{m_0+1}(\vartheta+2)^{m_0}}{(\vartheta+1)^{m_0+1} - (\vartheta+2)^{m_0} + (\vartheta+1)^{m_0} - \vartheta^{m_0+1}} + \frac{((\vartheta+1)^{m_0+1} - \vartheta^{m_0+1})\varepsilon_T}{((\vartheta+1)^{m_0}(\vartheta+2) - (\vartheta+2)^{m_0} - \vartheta^{m_0+1})C(s^*)}$$

where $\varepsilon_T \to 0$ as $T \to \infty$, $\vartheta = \lfloor \varphi_m \rfloor$ and φ_m is the unique nonnegative real solution to $(z+1)^m = z^{m+1}$, and $m_0 = \max_{e \in \mathcal{E}} m_e$

- The upper bund is tight.
- If f_e is known, pricing can be designed so that s^{ne}=s^* and POTA → 1
- When $m_e = 1$ for all edge e, the upper bound of POTA becomes 5/2, which coincides with the linear latency case

Analysis of Price of Total Anarchy

➢ Real Data Simulation

Consider the road network with two origin-destination pairs

- 500 players on each OD pair
- Fit linear latency to real data to get the values of all parameters

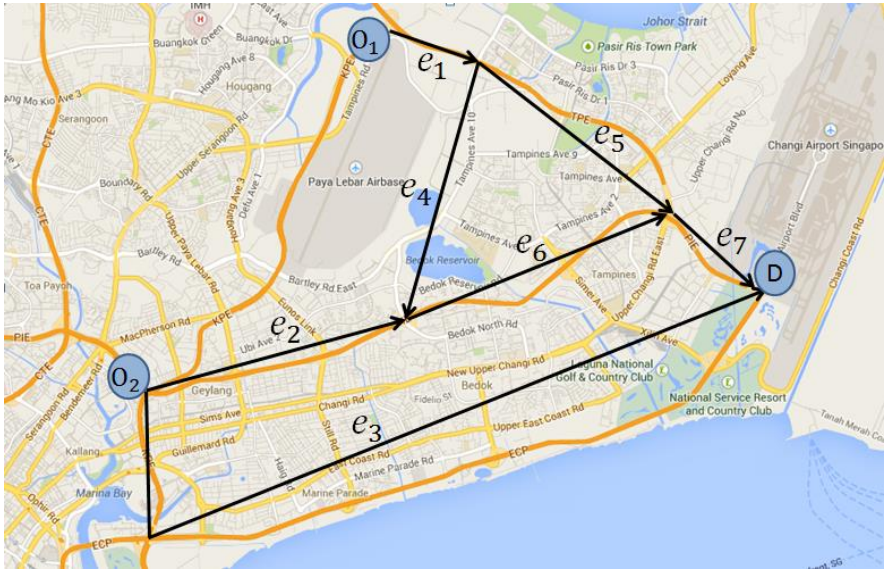


Fig. 14 Traffic network in east of Singapore

Analysis of Price of Total Anarchy

➢ Real Data Simulation

- The total latency of the socially optimal assignment is 118740.
- The average total latency without road price is 119340 and with the designed road price scheme is 118860, which indicates that the road price improves the average efficiency of the network.

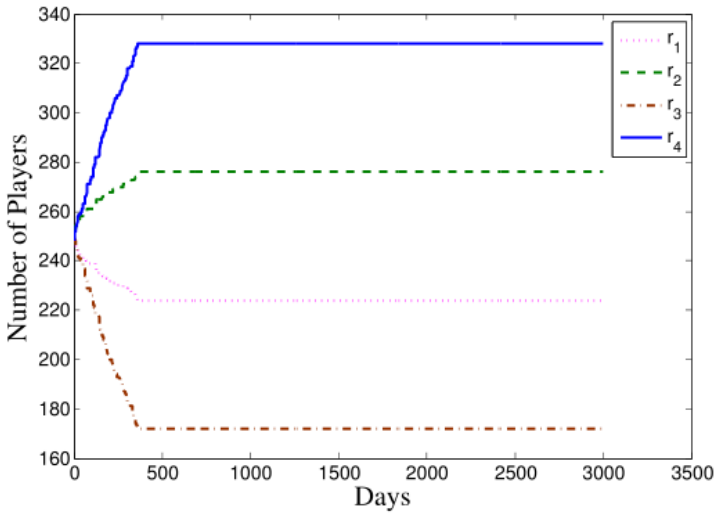


Fig. 15 Evolution of number of players on each route without price

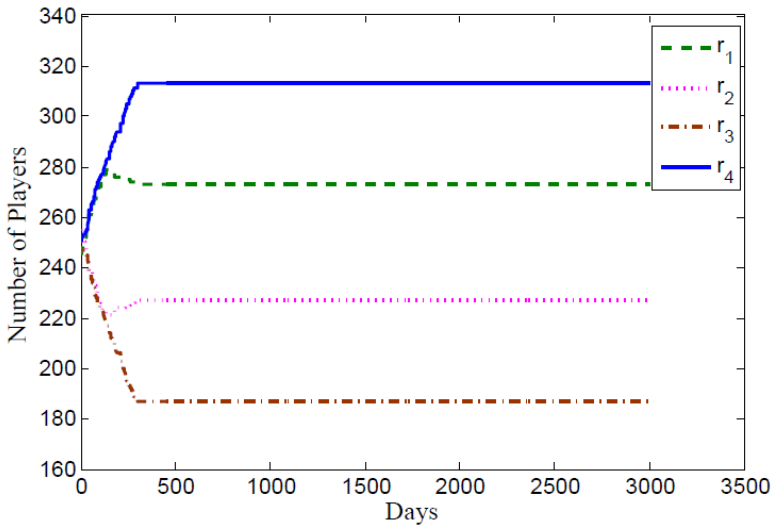


Fig. 16 Evolution of number of players on each route with designed road price

Outline

- Motivation

- Road Pricing Strategies: A Nash Equilibrium Perspective

  - Distributed consensus in non-cooperative games
  - Routing problem
  - Price of anarchy

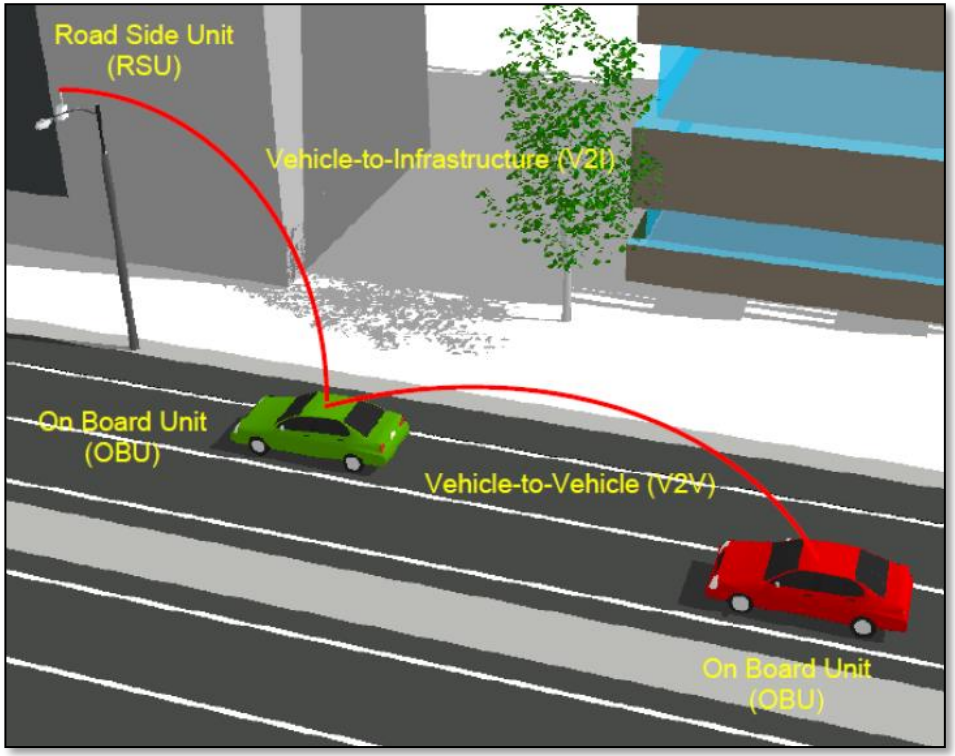
- **Conclusion and Opportunities**

Conclusion

- A consensus protocol is proposed to estimate the number of players on each resource
- The convergence property of Nash equilibrium is guaranteed in repeated games
- Several road pricing schemes were designed to improve the network efficiency in different models
- In traffic network with heterogeneous price-sensitive populations, we showed whether the social optimum flow can be achieved depends on the probability distribution of price sensitivity and the topology of the traffic network
- In traffic networks with multiple origin-destination pairs, we analyzed the POTA via smoothness argument

V2X Technology

- Vehicle to Vehicle (V2V) and Vehicle to Infrastructure (V2I) communication – V2X in short – allows cars to communicate wirelessly with cars using OBU, and with “access points” installed on traffic lights or lamp poles using RSU.



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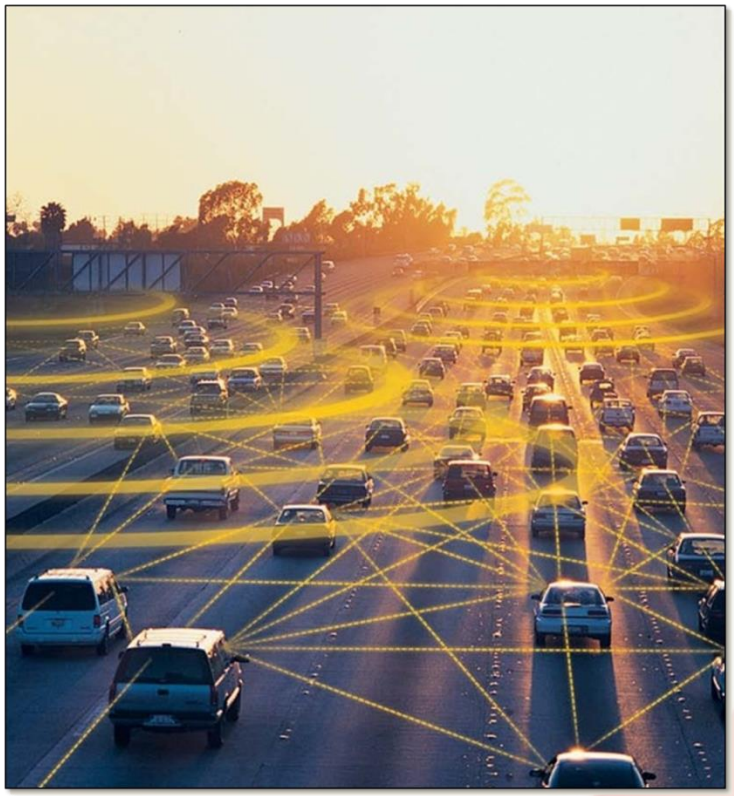
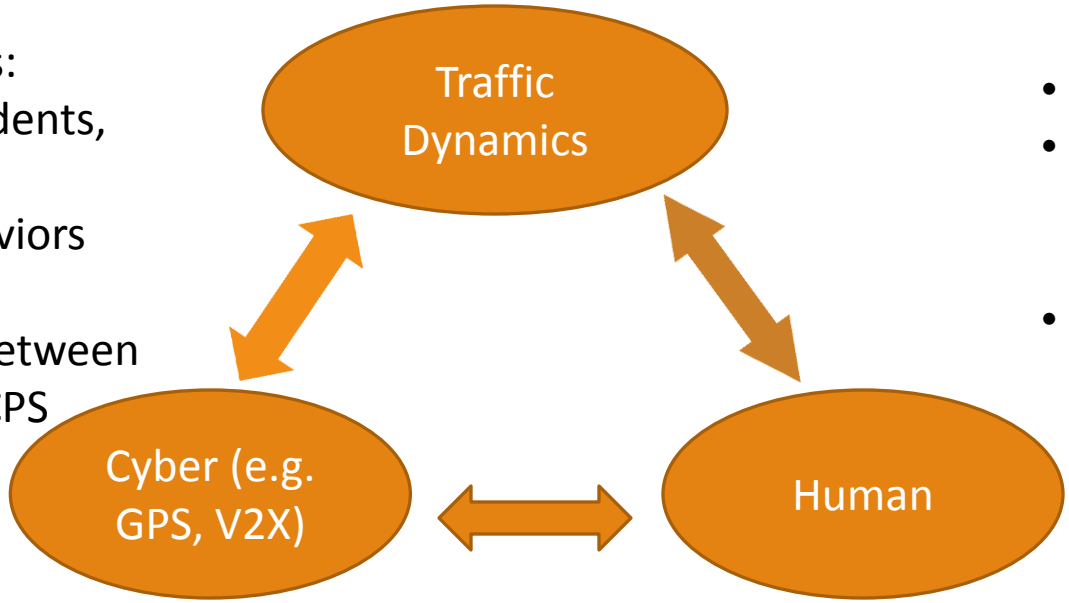


(Kapsch)

Opportunities & Challenges in Future Traffic Control Systems

- Filtering and distributed sensor fusion in the era of vehicle telematics and big data
- Situation aware distributed traffic light control
- New control techniques for road pricing design (e.g. Mean field games)
- Cyber-Physical-Human Systems:

- Uncertainties: Raining, accidents, events, etc
- Human behaviors analysis
- Interaction between human and CPS



- Big data analytics
- Distributed optimization and control
- System efficiency, robustness and resilience

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Thank you!