Existence, Convergence and Efficiency Analysis of Nash Equilibrium and Its Application to Traffic Networks

Lihua Xie
School of Electrical and Electronic Engineering
Nanyang Technological University, Singapore

(Joint work with Xuehe Wang)
Outline

- Motivation
- Related Work

- Road Pricing Strategies: A Nash Equilibrium Perspective
  - Distributed consensus in non-cooperative games
  - Routing problem
  - Price of anarchy

- Conclusion and Opportunities
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- Conclusion and Opportunities
**Motivation**

*Rapid Vehicle & Population Growth vs Limited Road Development*  
(Singapore 2002~2012)

Traffic control makes full utilization of existing infrastructure without road expansion

<table>
<thead>
<tr>
<th>Year</th>
<th>Vehicle</th>
<th>Population</th>
<th>Road</th>
</tr>
</thead>
<tbody>
<tr>
<td>2012</td>
<td>0.97 million</td>
<td>5.3 million</td>
<td>3,425 km</td>
</tr>
<tr>
<td>2030</td>
<td>1.2 million</td>
<td>6.9 million</td>
<td>3,700 km</td>
</tr>
<tr>
<td>Increase</td>
<td>~24%</td>
<td>~30%</td>
<td>~8%</td>
</tr>
</tbody>
</table>

Source: Land Transport Authority of Singapore
Motivation

➢ Traffic Congestion Problem

• Traffic congestion causes significant efficiency losses, wasteful energy consumption, and excessive air pollution
• It is difficult to enlarge the roadway capacity in major urban areas

➢ Congestion Losses

• In Europe, the external costs of road traffic congestion amount to 0.5% of Community GDP (White Paper—European Transport Policy for 2010)
• Sri Lanka loses 1.5% of the GDP (Rs 32 billion) due to traffic congestion (Business Times 2011)
• In UK, traffic congestion is costing the economy more than GBP 4.3 billion a year (Centre for Economics and Business Research 2012)
Traffic Control Systems

A control system to establish traffic regulations and their communications to the driver

<table>
<thead>
<tr>
<th>Component</th>
<th>Illustration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant</td>
<td>Dynamics of traffic flows</td>
</tr>
</tbody>
</table>
| Sensor       | (1) Loop detectors  
               | (2) Pneumatic tube counter  
               | (3) Cameras                 |
| Controller   | (1) Road price                 
               | (2) Traffic signals           
               | (3) Variable-message sign     |
| Actuator     | Driver’s action and/or decision                                             |
| Noise        | (1) Demand variations           
               | (2) Accidents, raining, fog                                               |
| Objective    | (1) Minimization of travel delay and/or number of stops (optimized performance)  
               | (2) Boundedness of vehicle queues (stability)                              |

Basic Control System:

- **Plant**: Dynamics of traffic flows
- **Sensor**:
  - (1) Loop detectors
  - (2) Pneumatic tube counter
  - (3) Cameras
- **Controller**:
  - (1) Road price
  - (2) Traffic signals
  - (3) Variable-message sign
- **Actuator**: Driver’s action and/or decision
- **Noise**:
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- **Objective**:
  - (1) Minimization of travel delay and/or number of stops (optimized performance)
  - (2) Boundedness of vehicle queues (stability)
Traffic Sensors

- Loop Detector
- Camera
- Pneumatic Tube Counter

Traffic Controllers

- Road Pricing
- Variable-message Sign
- Traffic Signals

Other controllers
- Signs and markings
- Car pooling programs
- Number plate auction
**ERP in Singapore**

- Electronic Road Pricing (ERP System)

  Alleviate traffic congestion by
  - affecting road users’ routing choices
  - refraining road users from travelling during peak hours

**Effect of Road Pricing (ERP system)**

- Traffic flow shoots up quickly just before the ERP begins
- Sharp decrease just at the time ERP begins
- Sharp rise just after ERP ends

*Fig. 1 The effect of ERP to number of vehicles on Anson Road in Singapore*
ERP2 in Singapore

• Expected to be implemented progressively from 2020 by the consortium of NCS Pte Ltd and Mitsubishi Heavy Industries Engine System Asia Pte Ltd, at a cost of S$556 million
• Based on Global Navigation Satellite System (GNSS) and Dedicated Short Range Communication (DSRC)
• Allow for more flexibility in managing traffic congestion through distance-based road pricing
• Provide services for motorists’ convenience, such as disseminating information on traffic advisories and facilitating e-payments
Objective

➢ Traffic Network

  • **Routing Problem**: Multiple origin-destination pairs and each origin-destination pair has several routes
  • **Trip Timing Problem**: Different departure time to avoid peak hour

➢ **Objective**: To develop road pricing strategies based on game theory and consensus control to manage traffic flows in an optimal manner and minimize traffic congestion

  • To estimate the mass behavior of all players via consensus control
  • To analyze the efficiency of Nash equilibrium and the performance in evolution of repeated games
  • To design dynamic pricing control to improve the efficiency of Nash equilibrium and the overall efficiency in repeated games for trip timing and routing problem
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    - Price of anarchy
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Related Work

➢ Game theory: deals with strategic interactions among multiple players, where each player tries to maximize his/her own utility (R. Gibbons, 1992).

Elements of a game

- **Players**: road users
- **Strategies**: route choices or trip timing available to each player \( i \), \( s_i \in R \)
- **Utility function**: \( U_i(s) \), where \( s = (s_i, s_{-i}) \)

Equilibrium concept (Nash)

- \( s^{ne} = (s_i^{ne}, s_{-i}^{ne}) \) is a Nash equilibrium if for any player \( i \),

\[
U_i(s_i^{ne}, s_{-i}^{ne}) = \max_{s_i \in R} U_i(s_i, s_{-i}^{ne})
\]

- None of players can improve his/her utility by a unilateral move
- Not always exist and may be inefficient
Related Work

- **Congestion game:** The utility/cost of each player depends on the strategy/resource it chooses and the number of players choosing the same strategy/resource (R. Rosenthal, 1973).

- **Potential game:**
\[ U_i(s^1_i, s_{-i}) - U_i(s^2_i, s_{-i}) = \Phi(s^1_i, s_{-i}) - \Phi(s^2_i, s_{-i}) \]

  - Guarantee the existence of Nash equilibrium (D. Monderer et al, 1996).
  - All players tend to jointly optimize the potential function.

Congestion game \(\xrightarrow{\text{(R. Rosenthal, 1973)}}\) Potential Game

Existence of Nash equilibrium
Related Work

- **Learning in games**: Allows players to adapt their strategies in response to the available information gathered over prior stages (D. Fudenberg et al, 1998).
  - Update perceptions of traffic conditions based on information broadcasted by government and/or obtained from other drivers through V2X
  - Inertia (intuitively): Some reluctance to change previous travel pattern

Driver’s decision process (from J.R. Marden etc. 2009)
Learning of Games: Fictitious Play

- Complete information of all utility functions is generally not available for individual player

- (Monderer 1996) Fictitious play: each player assumes that other players make decision independently according to observed empirical frequencies. The empirical frequencies generated by fictitious play of a potential game converge to a mixed strategy NE

- Shortcoming: when number of players is large, actual action for player $i$ at every stage is computationally infeasible since it depends on a mapping over a joint space

- (Marden et al. 2009) Joint strategy fictitious play: each player assumes that other players make decisions randomly and jointly according to joint empirical frequencies. It still ensures the convergence for potential games and reduces the computational burden of standard fictitious play

- Shortcoming: utility updating process is required for each player at every stage
Related Work

- **Discrete-time consensus protocol**: To estimate the number of players choosing each strategy for binary strategies case in inventory games (D. Bauso et al, 2009).
  - Exchange information with a set of neighbors.
  - Initial state of each agent – initial strategy of each player.
  - **Global objective**: Average-consensus – the percentage of players choosing each strategy.

- **Pricing schemes**: To improve the efficiency of Nash equilibrium.
  - The first-best pricing (marginal-cost pricing): the difference between the marginal social cost and the marginal private cost (M. J. Beckmann, 1967; M. Smith, 1979; H. Yang et al, 2004).
  - The second-best pricing: the location of the toll-gate, how much to charge, and the different impacts of the pricing schemes on different users (M. Marchand, 1968; T. Tsekeris et al, 2009).
  - Dynamic road pricing: vary according to real time road condition (R. B. Dial, 1999; T. Wongpiromsarn et al, 2012).
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  - Distributed consensus in non-cooperative games
  - Routing problem
  - Price of anarchy

➢ Conclusion and Opportunities
Distributed Consensus in Non-cooperative Congestion Games

- **Model Setup**

- **Notations:**
  - $\mathcal{R}_1 = \{r_1, r_2, \ldots, r_v\}$: public transportations.
  - $\mathcal{R}_2 = \{r_{v+1}, r_{v+2}, \ldots, r_{v+m}\}$: trip timing choices.
  - $s_i(k) = [s_{i1}(k), \ldots, s_{iv}(k), s_{iv+1}, \ldots, s_{iv+m}(k)]^T \in \mathbb{Z}^{v+m}$: player $i$’s choices, where $s_{ij} \in \{0, 1\}$ with $\sum_{j=1}^{v+m} s_{ij} = 1$.
  - $s(k) = \{s_1(k), s_2(k), \ldots, s_N(k)\}$: the action profile of all players.

Public transportation or private car and departure time?
Problem Formulation – Trip Timing Problem

- Utility of player $i$:

$$U_i(s_i(k), s_{-i}(k)) = \sum_{j=1}^{v} s_{ij}(k)p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(k)(f_j(n_{r_j}(k)) + h_i(r_j))$$ (1)

- **Lemma 1**: Congestion game with utility function (1) is a potential game with potential function:

$$\phi(s) = \sum_{j=1}^{v} \sum_{i=1}^{N} s_{ij}p_{ij} + \sum_{j=v+1}^{v+m} \sum_{d=1}^{n_{T_j}} f_j(d) + \sum_{j=v+1}^{v+m} \sum_{i=1}^{N} s_{ij}h_i(T_j)$$

- Existence of Nash equilibrium
Distributed Consensus in Non-cooperative Congestion Games

- If each player knows others’ strategies, \( \min_{s_i(k)} U_i(s_i(k), s_{-i}(k)) \)

- **Consensus Protocol**

  \[
  x_i(k) \in \mathbb{R}^{v+m}, \quad z_i(k) \in \mathbb{R}^{v+m}
  \]

  \[
  x_i(k + 1) = z_i(k) + \alpha(k) \sum_{j \in N_i(k)} L_{ij}(k) z_j(k)
  \]

  \[
  z_i(k) = x_i(k) + s_i(k) - s_i(k - 1) \quad \forall k \geq 1
  \]

  \[
  z_i(0) = x_i(0) = s_i(0)
  \]

  \( \alpha(k) \in (-1/\Delta(k), 0) \),
  \( \Delta(k) = \max_i (L_{ii}(k)) \)

Consensus protocol for **multiple strategies** case: alleviate the binary constraint of the inventory game (D. Bauso et al, 2009).

- **Lemma 2:** If no player changes strategy from stage \( t \) on and the collection of graphs over \( \tilde{T} \) is jointly connected, then \( x_i(t + \tilde{T}) \) is an estimate of the percentage of players choosing each resource at stage \( t \).
Estimate Percentage of Players with Broadcasting

- **Step 1**: at initial stage \( t = 0 \), every player picks up an action arbitrarily.
- **Step 2**: at stage \( t \geq 1 \), a system supervisor records \( n_{rt}(x(t-1)), \quad l = 1, 2, \ldots, M, \) and computes its weighted running average recursively as
  \[
  \bar{n}_{rt}(0) = n_{rt}(x(0)),
  \]
  \[
  \bar{n}_{rt}(t) = (1 - \lambda)\bar{n}_{rt}(t - 1) + \lambda n_{rt}(x(t - 1)),
  \]
  where \( \lambda \in (0, 1] \).
- **Step 3**: \( \bar{n}_{rt}(t) \) is broadcasted by system supervisor to all players, and player uses it for action selection.

Xiao et al., Average Strategy Fictitious Play with Application to Road Pricing, ACC 2013
Distributed Consensus in Non-cooperative Congestion Games

- Convergence of Nash Equilibrium

- Inertia:
  - If no opportunity for utility improvement, stay with previous strategy, i.e., \( s_i(k) = s_i(k - 1) \)
  - Otherwise, choose strategy which has maximal utility with probability \( \theta_i(k) \), where \( 0 < \theta_i(k) < 1 \)

- Theorem 1:

\[ G_N(k) : \text{jointly connected over each time interval } \tilde{T} \]

\[ U_i(s_i^1, s_{-i}) \neq U_i(s_i^2, s_{-i}) \]

Maintain strategies

Inertia

Consensus protocol

Converge to Nash equilibrium almost surely
Distributed Consensus in Non-cooperative Congestion Games

- **Dynamic Pricing**

  Marginal-cost (Pigovian tax, Mankiw, 2009): 
  \[
  p_j(n_{r_j}) = (n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j}))
  \]

  - Charge when players enter the road
  - Make players aware of the social cost instead of private cost

  Utility of player \(i\):
  \[
  U_i(s_i(k), s_{-i}(k)) = \sum_{j=1}^{v} s_{ij}(k)p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(k)(f_j(n_{r_j}(k)) + h_i(r_j) - p_j(n_{r_j}(k)))
  \]

  Potential function:
  \[
  \phi(s) = \sum_{i=1}^{N} \left( \sum_{j=1}^{v} s_{ij}p_{ij} + \sum_{j=v+1}^{v+m} s_{ij}(f_j(n_{r_j}) + h_i(r_j)) \right)
  \]

  - All players jointly optimize the social utility

Some sort of social optimal
Case Study of Singapore

- Travel time formula introduced by P. Patriksson, 1994:

\[ d(n_{r_j}) = t_0 \cdot \left( 1 + 0.15 \frac{n_{r_j}}{q_0} \right)^{m_\alpha} \]

Free-flow travel time: travel time at zero flow

Practical capacity: the ow from which the travel time will increase very rapidly if the ow is further increased

Data fitting:
- \( t_0 = 0.0998 \)
- \( m_\alpha = 3.977 \)
- \( q_0 = 1357 \)

Fig. 2 The relationship between traffic flow and travel time
Distributed Consensus in Non-cooperative Congestion Games

Case Study of Singapore

- Average speed: \( f_j(n_{r_j}) = \frac{L_0}{d(n_{r_j})} \)
- Deviation from preferred departure time: \( h_i(r_j) = \alpha_i |r_j - \hat{T}_i| \)

Overall utility without price: 95219
Overall utility with price: 96781

\( N = 2000, \alpha_i \in [-1.5, -0.5] \)

May still cause traffic congestion

Fig. 3 Evolution of number of vehicles on each choice without price.
Fig. 4 Evolution of number of vehicles on each choice with price.
Distributed Consensus in Non-cooperative Congestion Games

- **Dynamic Pricing**

  - **Case without Public Transportation**

    Entropy term: \( \sum_{j=v+1}^{v+m} q_j \log q_j \)

    - Reach minimum value only when \( q_{v+1} = \ldots = q_{v+m} \)

  Road pricing:

  \[
  \tilde{p}_j(n_{r_j}) = (n_{r_j} - 1)(f_j(n_{r_j} - 1) - f_j(n_{r_j})) + \frac{w}{N} \log\left(\frac{n_{r_j}}{n_{r_j} - 1}\right) + \frac{w}{N} \log(n_{r_j})
  \]

  Potential function:

  \[
  \tilde{\phi}(s) = \sum_{i=1}^{N} \left( \sum_{j=v+1}^{v+m} s_{ij}(f_j(n_{r_j}) + h_i(r_j)) \right) - w \sum_{j=v+1}^{v+m} q_j \log q_j
  \]

  - Action profiles generated by the utility function converges to pure NE a.s.
  - As \( w \) goes to infinity, the disparity in the number of players choosing each resource will vanish
Case Study of Singapore

- Case without Public Transportation

Fig. 5 Evolution of number of vehicles on each choice without price

Fig. 6 Evolution of number of vehicles on each choice with marginal cost

Fig. 7 Evolution of number of vehicles on each choice with price function with entropy term

Set $w = 200000$
➢ Price Sensitivity

Each road user has a price sensitivity $\beta$ which may be different for different road users

- $\{\beta_1, \ldots, \beta_M\}$: set of price sensitivities
- $\{p_1, \ldots, p_M\}$: the corresponding distribution

<table>
<thead>
<tr>
<th>Mode</th>
<th>Value of Time (¢/min)</th>
<th>Price sensitivities (min/¢)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
<td>4.0</td>
<td>0.25</td>
</tr>
<tr>
<td>Motorcycle</td>
<td>2.8</td>
<td>0.36</td>
</tr>
<tr>
<td>Taxi</td>
<td>5.1</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Price sensitivities (from 2004 Stated Preference Survey)
Road Pricing for Routing Problem

➢ Traffic network

- Each edge $e$ possesses a linear latency function (T. Roughgarden et al, 2002):
  \[ l_e(f_e) = d_e f_e + c_e \]
  - $d_e$: additional travel time due to one more player on link $e$
  - $c_e$: free-flow travel time on link $e$

- Latency of each route $r$:
  \[ l_r(f_r) = \sum_{e \in r} l_e(f_e) \]

- Total latency of the network:
  \[ L(f) = \sum_{r \in R} f_r \cdot l_r(f_r) \]

- Cost of route $r$ for group with price sensitivity $\beta$:
  \[ J_{r,\beta}(f) = \sum_{e \in r} (l_e(f_e) + \beta \rho_e(f_e)) \]
  - $\rho_e(f_e)$: Toll on each edge $e$

Social optimal flow:
\[ f^* = \arg \inf_f L(f) \]
Nash flow (Nash equilibrium): A flow $f^{ne}$ is called a Nash flow, if for any $\beta_j$ and $r_1, r_2$,

$$f^{ne}_{r_1,\beta_j} > 0, f^{ne}_{r_2,\beta_j} > 0 \Rightarrow J_{r_1,\beta_j}(f^{ne}) = J_{r_2,\beta_j}(f^{ne}),$$

$$f^{ne}_{r_1,\beta_j} > 0, f^{ne}_{r_2,\beta_j} = 0 \Rightarrow J_{r_1,\beta_j}(f^{ne}) \leq J_{r_2,\beta_j}(f^{ne}).$$

- All players choose minimum-cost routes
- For each group, the costs of the selected routes are the same

Price of anarchy (POA): The worst possible ratio between the total latency of a Nash flow and that of the optimal flow:

$$\text{POA} = \sup_{f^{ne}} \frac{L(f^{ne})}{L(f^*)}$$

POA: the smaller, the better

- The POA can be arbitrarily large
- It is proved by T. Roughgarden et al. (2002) that POA is bounded by $4/3$ if the network has linear latency functions
- Sensitive to users’ price sensitivities (Brown and Marden, CDC 2014). Nash flow could be as much as 33% more congested than an optimal flow for unknown sensitivities.

Road Pricing for Routing Problem
Method of Reducing POA

- **Marginal-cost**: The difference between the marginal social cost and the marginal private cost
  - Make player aware of the losses that it imposes on other players
  - Totally eliminate the efficiency losses due to selfish behavior of players, i.e., POA=1
  - Only apply to homogeneous case when players have the same price sensitivity
- **Scaled marginal cost**: A variant of marginal cost which can be applied to heterogeneous case
Design of Road Pricing

Same price sensitivity $\beta$

Homogeneous players

Marginal cost toll on each edge $e$

$$\rho_e(f_e) = f_e l'_e(f_e)$$

Nash flow $\rightarrow$ Social optimal flow

Heterogeneous players

Scaled marginal cost toll on edge $e$

$$\rho_e(f_e) = \mu f_e l'_e(f_e)$$

Goal:

$$L(f^{ne}(\mu^*)) = \inf_{\mu \geq 0} L(f^{ne}(\mu))$$

Remark: Homogeneous case with uncertain $\beta \in [\beta_L, \beta_U], \mu^* = \frac{1}{\sqrt{\beta_L \beta_U}}$ (Brown and Marden, 2014)

Road Pricing for Routing Problem

- **Theorem 2**: The optimal POA depends on the distribution of price sensitivities and the topology of the network.

- Special case – Two groups of players in a two routes network: always exists $\mu^*$ such that $POA=1$
- $POA$ depends on the distribution of price sensitivity, road network topology and parameters of latency functions.

![Diagram](image)
Analyze of Price of Anarchy

• Best POA Algorithm: find $\mu^*$ that minimizes the POA for any distribution of price sensitivity and networks with one origin-destination pair.

Road Pricing for Routing Problem

**Input:** Network parameters (e.g., $d_e, c_e$) and network topology, total flow $F$, price sensitivities $B$ and its probability distribution $P$

**Output:** Best POA

Calculate the social optimal flow $f^*$;

For $k_1 = 1 : N$

Group $\beta_1$ choose a sequential routes $r_{k_1}, ..., r_{k_N}$ with flow $f_{r_{k_1}, \beta_1} > 0, ..., f_{r_{k_N}, \beta_1} > 0$;

... 

For $k_{j+1} = 1 : k_j$

Group $\beta_{j+1}$ choose a sequential routes $r_{k_{j+1}}, ..., r_{k_{k_j}}$ with flow $f_{r_{k_{j+1}, \beta_{j+1}}} > 0, ..., f_{r_{k_{k_j}, \beta_{j+1}}} > 0$;

... 

For $k_M = k_{M-1} - 1 : k_{M-1}$

Group $\beta_M$ choose a sequential routes $r_1, ..., r_{k_M}$ with flow $f_{r_1, \beta_M} > 0, ..., f_{r_{k_M}, \beta_M} > 0$;

Solve the equations set

Return Nash flow $f^{ne}(\mu)$;

Find $\mu^*$ by optimizing $L(f^{ne}(\mu))$;

... 

end

end

Return the largest value of $\frac{L(f^{ne}(\mu^*))}{L(f^*)}$

Remarks:

• The results on the non-cooperative games can be extended to include combined routing and trip timing
• Can be extended to multiple S-D case.
Real Data Simulation

4 groups of road users with total flow 200.

<table>
<thead>
<tr>
<th>Group</th>
<th>Price sensitivity (min/¢)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
<td>0.25</td>
</tr>
<tr>
<td>Motorcycle</td>
<td>0.36</td>
</tr>
<tr>
<td>Taxi</td>
<td>0.20</td>
</tr>
<tr>
<td>Bicycle</td>
<td>0</td>
</tr>
</tbody>
</table>

Price sensitivities for each group (from 2004 Stated Preference Survey)

<table>
<thead>
<tr>
<th>Group</th>
<th>Vehicle Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
<td>0.7239</td>
</tr>
<tr>
<td>Motorcycle</td>
<td>0.1884</td>
</tr>
<tr>
<td>Taxi</td>
<td>0.0281</td>
</tr>
<tr>
<td>Bicycle</td>
<td>0.0596</td>
</tr>
</tbody>
</table>

Vehicle distribution (from Singapore land transport statistics in brief 2005)
Real Data Simulation

Consider the East Road Network of Singapore with 4 routes.

Fig. 8 Traffic network in the east of Singapore

Fig. 9 Linear fitting of certain link

\[ d_e = 0.0932 \]
\[ c_e = 5.5409 \]

\[ l_e = 0.0932 f_e + 5.5409 \]
Optimal road price: $\mu^* = 3.394$ such that $\text{POA} > 1$, i.e., the total latency of the Nash flow can’t achieve the total latency of the socially optimal flow.

- Total latency without road price: 35950.711
- Total latency with road price: 35922.899
- Total latency of the optimal flow: 35922.807

Fig. 10 The relationship between $\mu^*$ and POA for road network
Real Data Simulation

Consider the road network with 3 routes in CBD.

Fig. 11 Simple traffic network in Central Business District (CBD)

Fig. 12 Linear fitting of certain link

\[ d_e = 0.0327 \]
\[ c_e = 1.4825 \]

\[ l_e = 0.0327f_e + 1.4825 \]
Real Data Simulation

Optimal road price: $\mu^* = 4$ such that POA=1, i.e., the total latency of the Nash flow is equal to the total latency of the socially optimal flow.

- Total latency without road price is 2472.8.
- Total latency with road price is 2457.8.

Fig. 13 The relationship between $\mu^*$ and POA for road network
**Problem Formulation**

- Each origin-destination pair: fixed number of players

- Each player \( i \) selects a route \( s_i \) from his/her strategy set \( S_i \) (players in same origin-destination pair have the same strategy set)

- Latency \( l_e(f_e) \) on each edge is associated with the number of players on this edge

- Cost received by player \( i \) is

\[
C_i(s) = \sum_{e \in s_i} l_e(f_e)
\]

- **Nash equilibrium**: for each player \( i \),

\[
C_i(s_{ne}) = \min_{s_i \in S_i} C_i(s_i, s_{ne}^{ne}) \quad \text{May be inefficient}
\]

Congestion game: Nash equilibrium always exists

---

No-regret

• Consider a sequence of strategies \(s(1), s(2), \ldots, s(T)\), define

\[
\delta_i(s(t)) = C_i(s(t)) - \min_{s_i \in S_i} C_i(s_i, s_{-i}(t))
\]

• A strategy sequence exhibits almost sure \(\epsilon\)-no-regret iff

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \delta_i(s(t)) < \epsilon \quad \text{a.s.}
\]

If the sequence exhibits almost surely \(\epsilon\)-no-regret for any \(\epsilon > 0\), then it is said to exhibit almost sure no-regret.

• For strategy profiles generated by best response with inertia, there exists a \(\epsilon_T > 0\) such that

\[
\frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \delta_i(s(t)) = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_{i=1}^{N} C_i(s(t)) - \sum_{i=1}^{N} \min_{s_i \in S_i} C_i(s_i, s_{-i}(t)) \right) \leq \epsilon_T
\]

\(\epsilon_T \to 0\) as \(T \to \infty\) \quad \text{Almost sure no-regret}
Analysis of Price of Total Anarchy

- **Price of Total Anarchy**

  - The **price of total anarchy (POTA)** is defined as the worst-ratio between the average total latency and the total latency of the optimal assignment

    \[
    \text{POTA} = \frac{\frac{1}{T} \sum_{t=1}^{T} C(s(t))}{C(s^*)}
    \]

  - A cost-minimization game is \((\lambda, \mu)-\text{smooth}\) if for every two strategies \(s^1\) and \(s^2\):

    \[
    \sum_{i=1}^{N} C_i(s_i^2, s_{-i}^1) \leq \lambda \cdot C(s^2) + \mu \cdot C(s^1)
    \]

    \[
    \text{(2)}
    \]

  Specially, for a Nash equilibriun \(s^{ne}\) and the optimal strategy \(s^*\):

    \[
    C(s^{ne}) = \sum_{i=1}^{N} C_i(s^{ne}) \leq \sum_{i=1}^{N} C_i(s_i^*, s_{-i}^{ne}) \leq \lambda \cdot C(s^*) + \mu \cdot C(s^{ne})
    \]

  If a game is \((\lambda, \mu)-\text{smooth}\) with \(\lambda>0\) and \(\mu<1\), then \(\text{POA} \leq \frac{\lambda}{1-\mu}\).
Analysis of Price of Total Anarchy

- **Price of Total Anarchy**

  - Define \( \delta_i^*(s(t)) = C_i(s(t)) - C_i(s^*, s_{-i}(t)) \), obviously, \( \delta_i(s(t)) \geq \delta_i^*(s(t)) \)

  - In smooth game, at stage \( t \),

    \[
    \frac{1}{T} \sum_{t=1}^{T} C(s(t)) \leq \frac{\lambda}{1 - \mu} C(s^*) + \frac{1}{1 - \mu} \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} \delta_i^*(s(t))
    \]

  - The robust POA of a cost-minimization game is

    \[
    \inf \left\{ \frac{\lambda}{1 - \mu} : (\lambda, \mu) \text{ such that the game is } (\lambda, \mu) - \text{smooth} \right\} \quad \mu < 1
    \]

  *Theorem 3*: The action profiles generated by best response with inertia will converge a NE a.s. in finite time and if the robust POA of the game is \( \rho \), then there exists a \( \varepsilon_T > 0 \) such that

    \[
    \text{POTA} \leq \rho + \frac{\varepsilon_T}{(1 - \mu) C'(s^*)}
    \]

    almost surely, where \( \varepsilon_T \to 0 \) as \( T \to \infty \)
Traffic Network with Linear Latency

Set the latency on each edge as \( l_e(f_e) = a_e f_e + b_e \)

- The networks with linear latency functions is a \((\frac{5}{3}, \frac{1}{3})\)-smooth game (G. Christodoulou et al, 2006)

**Lemma 3:** For action profiles generated by best response with inertia, POTA \( \leq \frac{5}{2} + \frac{3\varepsilon_T}{2C(s^*)} \) almost surely, where \( \varepsilon_T \to 0 \) as \( T \to \infty \).

- To improve the overall efficiency, motivated by the form of marginal-cost price, the road price is designed as

\[
p_e(f_e) = w_e(f_e - 1)
\]

\( w_e : \) Nonnegative constant to be designed

**Lemma 4:** For action profiles generated by best response with inertia and POTA \( \leq \frac{49}{24} + \frac{\tau_T}{C(s^*)} \) almost surely, where \( \tau_T \to 0 \) as \( T \to \infty \).

**Remark:** By charging the road price \( p_e(f_e) = a_e(f_e - 1) \), the upper bound of the POTA decreases compared to that without road price.
Traffic Network with Nonlinear Latency

• Latency on each edge (Patriksson, 1994)

\[ l_e(f_e) = t_e \cdot (1 + d_e(f_e/c_e)^{m_e}) \]

Lemma 5: For congestion games with nonlinear latency, if the action profiles are generated by best response with inertia, then,

\[
POTA \leq \frac{(\vartheta + 1)^{2m_0+1} - \vartheta^{m_0+1}(\vartheta + 2)^{m_0}}{(\vartheta + 1)^{m_0+1} - (\vartheta + 2)^{m_0} + (\vartheta + 1)^{m_0} - \vartheta^{m_0+1}}
+ \frac{((\vartheta + 1)^{m_0+1} - \vartheta^{m_0+1})\varepsilon_T}{((\vartheta + 1)^{m_0}(\vartheta + 2) - (\vartheta + 2)^{m_0} - \vartheta^{m_0+1})C(s^*)}
\]

where \( \varepsilon_T \to 0 \) as \( T \to \infty \), \( \vartheta = \lceil \varphi_m \rceil \) and \( \varphi_m \) is the unique nonnegative real solution to \( (z+1)^m = z^{m+1} \), and \( m_0 = \max_{e \in \mathcal{E}} m_e \)

• The upper bound is tight.
• If \( f_e \) is known, pricing can be designed so that \( s^{ne} = s^* \) and POTA \( \to 1 \)
• When \( m_e = 1 \) for all edge \( e \), the upper bound of POTA becomes \( 5/2 \), which coincides with the linear latency case
Real Data Simulation

Consider the road network with two origin-destination pairs

- 500 players on each OD pair
- Fit linear latency to real data to get the values of all parameters

Fig. 14 Traffic network in east of Singapore
Real Data Simulation

- The total latency of the socially optimal assignment is 118740.
- The average total latency without road price is 119340 and with the designed road price scheme is 118860, which indicates that the road price improves the average efficiency of the network.

Fig. 15 Evolution of number of players on each route without price

Fig. 16 Evolution of number of players on each route with designed road price
Outline

➢ Motivation

➢ Road Pricing Strategies: A Nash Equilibrium Perspective
  - Distributed consensus in non-cooperative games
  - Routing problem
  - Price of anarchy

➢ Conclusion and Opportunities
Conclusion

- A consensus protocol is proposed to estimate the number of players on each resource
- The convergence property of Nash equilibrium is guaranteed in repeated games
- Several road pricing schemes were designed to improve the network efficiency in different models
- In traffic network with heterogeneous price-sensitive populations, we showed whether the social optimum flow can be achieved depends on the probability distribution of price sensitivity and the topology of the traffic network
- In traffic networks with multiple origin-destination pairs, we analyzed the POTA via smoothness argument
V2X Technology

- Vehicle to Vehicle (V2V) and Vehicle to Infrastructure (V2I) communication – V2X in short – allows cars to communicate wirelessly with cars using OBU, and with “access points” installed on traffic lights or lamp poles using RSU.

(Cohda)

(Kapsch)
Opportunities & Challenges in Future Traffic Control Systems

- Filtering and distributed sensor fusion in the era of vehicle telematics and big data
- Situation aware distributed traffic light control
- New control techniques for road pricing design (e.g. Mean field games)
- Cyber-Physical-Human Systems:
  - Uncertainties: Raining, accidents, events, etc
  - Human behaviors analysis
  - Interaction between human and CPS
  - Big data analytics
  - Distributed optimization and control
  - System efficiency, robustness and resilience
Thank you!